THE ZERO-TYPE PROPERTY AND MIXING OF BERNOULLI SHIFTS

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Abstract. We prove that every non-singular Bernoulli shift is either zero-type or there is an equivalent invariant stationary product probability. We also give examples of a type III_1 Bernoulli shift and a Markovian flow which are power weakly mixing and zero type.

Introduction. This article deals with the concept of zero-type for invertible non-singular transformations T of the standard probability space $(X, \mathcal{B}, m)$.

A non-singular transformation without an absolutely continuous invariant probability is called zero-type, sometimes also called mixing, if its Koopman operator is mixing, meaning that it's maximal spectral type is a Rajchman measure. This generalises the notion of zero-type introduced by Hajian and Kakutani [HK]. The name mixing comes from the fact that in the case of probability preserving transformations the classical notion of mixing is equivalent to mixing of the Koopman operator restricted to $L^2_0(X,m) = L^2(X,m) \ominus \mathbb{C}$. In the interplay between non singular ergodic theory and infinitely divisible processes, the zero type transformations are associated with the classical notion of mixing of infinitely divisible processes, see [RS] or [Ro].

In the first part we prove that every non-singular Bernoulli shift is either zero-type or there is an equivalent invariant product probability. Thus the Hamachi shift [Ham] and the type III_1 shift given in [Kos] are examples of conservative, ergodic, zero-type transformations which do not posses an m equivalent σ-finite invariant measure, also known as type III and zero type transformations. For another construction of zero type, type III transformations see [Da1, Theorem 0.3].

A non singular transformation T is weakly mixing if $S \times T$ is ergodic for every ergodic probability preserving transformation S. It follows that if T is a probability preserving weakly mixing transformation then $T \times \cdots \times T$ ($k-$times) is ergodic for every $k \in \mathbb{N}$. In general this is not true, as there exist weakly mixing transformations such that $T \times T$ is not ergodic. In an attempt to understand the notion of weak mixing for non singular transformations the concept of power weakly mixing transformations was introduced. T is called *power weakly mixing* if for any $l_1, l_2, ..., l_k \in \mathbb{Z}\backslash\{0\}$,

$$\left(X^k, P^{\otimes k}, T^{l_1} \times T^{l_2} \times \cdots \times T^{l_k}\right)$$

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is an ergodic automorphism where

$$P^{\otimes k} = \underbrace{P \otimes P \otimes \cdots \otimes P}_{k-\text{times}}.$$

A weaker notion is that of infinite ergodic index which means that for every $k \in \mathbb{N}$, the k-fold product of T is an ergodic automorphism. In [AFS] it was shown that Chacon's non singular type III_λ transformation, $0 < \lambda \leq 1$, is a power weakly mixing transformation. In [Da2] a construction of a transformation which is of infinite ergodic index but not power weakly mixing was given. Later in [DP] a flow was constructed such that all times are of infinite ergodic index. These constructions use the cutting and stacking method which usually doesn't give a zero type transformation.

We give a construction of a Bernoulli shift which is power weakly mixing and type III_1. By the first part it is also zero type. Finally a continuous time flow $\{\phi_t\}_{t\in\mathbb{R}}$ is given such that for every $a_1, a_2, ..., a_n \in \mathbb{R}$,

$$\phi_{a_1} \times \phi_{a_2} \times \cdots \times \phi_{a_n}$$

is ergodic. This flow is the time shifts of a continuous time Markov Chain.

Preliminaries: Let $(X, \mathcal{B}, P)$ be a probability space. An invertible measurable transformation $T : X \to X$ is said to be *non-singular* with respect to P if it preserves the $P-$null sets. i.e for every $A \in \mathcal{B}$, $P(A) = 0$ if and only if

$$P \circ T(A) := P(TA) = 0.$$

A measure m on X is said to be $T-$*invariant* if for every $A \in \mathcal{B}$,

$$P \circ T(A) = P(A).$$

If T is non-singular, then for every n, $P \circ T^n$ is absolutely continuous with respect to P. By the Radon-Nikodym theorem there exist measurable functions $\frac{dP\circ T^n}{dP} \in L_1(X, P)_+$ such that for every $A \in \mathcal{B}$,

$$P \circ T^n(A) = \int_A \left(\frac{dP \circ T^n}{dP}\right) dP.$$

Denote $(T^n)'(x) := \frac{dP\circ T^n}{dP}$.

A set $W \in \mathcal{B}$ is called *wandering* if $\{T^n W\}_{n=-\infty}^{\infty}$ are disjoint. As in [Aar, p.7] denote by $\mathfrak{D}$ the measurable union of all wandering sets for T, this set is $T-$invariant. Its complement is denoted by $\mathfrak{C}$.

Definition. We call $(X, \mathcal{B}, P, T)$ *dissipative* if $\mathfrak{D} = X$. If $\mathfrak{C} = X$ then $(X, \mathcal{B}, P, T)$ is said to be *conservative.*

We will use the following version of Hopf's decomposition theorem which says that the conservative and dissipative parts can be separated in the following way.

Theorem I. *For every non-singular transformation T of the probability space $(X,\mathcal{B},P)$ there exists a decomposition $X=\mathfrak{D}\cup\mathfrak{C}$ such that $(X,\mathcal{B},P|_{\mathfrak{D}},T)$ is dissipative and $(X,\mathcal{B},P|_{\mathfrak{C}},T)$ is conservative. Furthermore*

$$\sum_{k=0}^{\infty}\left(T^{-n}\right)'(x)<\infty \text{ a.e. } x\in\mathfrak{D}$$

and

$$\sum_{k=0}^{\infty}\left(T^{-n}\right)'(x)=\infty \text{ a.e. } x\in\mathfrak{C}.$$

The Hellinger Integral and definition of the Zero Type Property. The Koopman operator $U:L_2\left(X,\mathcal{B},P\right)$ is then defined by

$$Uf(x)=\sqrt{T'(x)}f\circ T(x).$$

It is a unitary operator and by the chain rule for the Radon-Nikodym derivatives for every $n\in\mathbb{Z}$,

$$U^n f=\sqrt{(T^n)'}f\circ T^n.$$

Definition. A transformation is *Non-Singular zero-type* (NS zero type) if the maximal spectral type $\sigma_T\in\mathcal{P}\left(\mathbb{T},\mathcal{B}\left(\mathbb{T}\right)\right)$ of U_T is a Rajchman measure. That is its Fourier coefficients $\hat{\sigma}_T(n)$ tend to 0 as $n\to\infty$. and so for every $f\in L_2(X,P)$,

$$\int_X f\cdot U^n f dP\to 0 \text{ as } n\to\infty.$$

Remark. By looking at the Koopman operators it is seen that the zero-type property depends only on the equivalence class of P.

Remark. In the case when $(X,\mathcal{B},m,T)$ is a σ-finite measure preserving transformation Krengel and Sucheston [KrS] have showed that mixing of the Koopman operator is equivalent to the Hajian and *Kakutani zero-type condition* [HK] which states that for every $A\in\mathcal{B}$ with $m(A)<\infty$,

$$\lim_{n\to\infty} m\left(A\cap T^{-n}A\right)=0.$$

Definition. Let P,Q be two probability measures on X. The *Hellinger Integral* (see [Hel] or [Kak]) is then defined as

$$\rho(P,Q)=\int_X\sqrt{\frac{dP}{dm}}\cdot\sqrt{\frac{dQ}{dm}}\,dm$$

where m is any finite measure on X such that $P \ll m$, $Q \ll m$. In the special case where $P \ll Q$ we can take $m = Q$ and have

$$\rho(P,Q) = \int_X \sqrt{\frac{dP}{dQ}}\, dQ.$$

The function $\rho(\cdot,\cdot)$ measures the amount of singularity of P with respect to Q. This function satisfies that for every $P, Q \in \mathcal{P}(X)$,$0 \le \rho(P,Q) \le 1$. Also $\rho(P,Q) = 0$ if and only if P is singular with respect to Q.

The proof of the following proposition is standard.

Proposition 1. *Let T be a non-singular transformation of the probability space $(X, \mathcal{B}, P)$. The following are equivalent.*
(i) $\lim\limits_{n\to\infty} \rho\left(P, P \circ T^n\right) = 0$.
(ii) $(T^n)' \stackrel{P}{\longrightarrow} 0$.
(iii) σ_T *is a Rajchman measure.*

1. Bernoulli shifts are zero-type or mixing

Let $\mathbb{X} = \{0,1\}^{\mathbb{Z}}$ and T be the left shift action on $\mathbb{X}$, that is

$$(Tw)_i = w_{i+1}.$$

Denote the cylinder sets by

$$[b]_k^l = \{w \in \mathbb{X} : \ \forall i = k, ..., l, \ w_i = b_i\}.$$

A measure $P = \prod\limits_{k=-\infty}^{\infty} P_k \in \mathcal{P}(\mathbb{X})$ is called a product measure if for every $k < l$, and for every cylinder $[b]_k^l$,

$$P\left([b]_k^l\right) = \prod_{j=k}^{l} P_j\left(\{b_j\}\right).$$

We will say that a product measure P is *non-singular* if the shift is non-singular with respect to P.

For two product probability measures P, Q and $N \in \mathbb{N} \cup \{\infty\}$ define

$$\begin{aligned} d_N(P,Q) &:= \sum_{k=-N}^{N} \left\{ \left(\sqrt{P_k(\{0\})} - \sqrt{Q_k(\{0\})} \right)^2 . \right. \\ &\left. + \left(\sqrt{P_k(\{1\})} - \sqrt{Q_k(\{1\})} \right)^2 \right\} \end{aligned}$$

Notice that $d_N(P,Q) \uparrow d_\infty(P,Q)$ as $N \to \infty$. Set $d(P,Q) := d_\infty(P,Q)$

The following lemma is a direct consequence of Kakutani's Theorem on equivalence of product measures[Kak].

Lemma 2. *Let* $P = \prod_{k=-\infty}^{\infty} P_k$ *be a product measure. Then*

(1) For any two product measures P, Q*,*

$$d(P,Q) \propto -\log \rho(P,Q).$$

(2) P *is non-singular if and only if*

$$d(P, P \circ T) < \infty. \tag{1.1}$$

(3) The shift is NS zero-type if and only if

$$\lim_{n\to\infty} d(P, P \circ T^n) = \infty.$$

Theorem 3. *Let* $P = \prod_{k=-\infty}^{\infty} P_k$ *be a non-singular product measure. Then either there exists a shift invariant* P*-equivalent probability or the shift* $(\mathbb{X}, \mathcal{B}(\mathbb{X}), P, T)$ *is NS zero-type. Therefore a non-singular shift is either mixing in the probability preserving sense or mixing in the non-singular sense.*

Theorem 1 follows from lemmas 4 and 5.

Lemma 4. *Let* P *be a non-singular product measure on* $\{0,1\}^{\mathbb{Z}}$ *such that*

$$\exists \lim_{k\to-\infty} P_k = (p, 1-p) := \mu_p.$$

Denote by $Q = \prod_{k=-\infty}^{\infty} \mu_p$*. Then if* $P \perp Q$ *then* $(\mathbb{X}, \mathcal{B}(\mathbb{X}), P, T)$ *is of NS-zero type. Else* Q *is a* P*-equivalent shift-invariant probability measure.*

Proof. Assume that $P \perp Q$. Then by Kakutani's theorem

$$d(P,Q) = \infty.$$

By claim 2 its enough to show that $\lim_{n\to\infty} d\left(P, P\circ T^{n}\right) = \infty$.

Let $M>0$. Since $d(P,Q)=\infty$ there exists a $N\in\mathbb{N}$ such that

$$d_N(P,Q) > M.$$

For every $n\in\mathbb{N}$,

$$\begin{aligned} d\left(P, P\circ T^{n}\right) \;\; &\geq \;\; \sum_{k=-N}^{N}\left\{\left(\sqrt{P_k\left(\{0\}\right)}-\sqrt{P_{k-n}\left(\{0\}\right)}\right)^2\right. \\ &\qquad \left. +\left(\sqrt{P_k\left(\{1\}\right)}-\sqrt{P_{k-n}\left(\{1\}\right)}\right)^2\right\} \end{aligned}$$

Therefore since $\lim_{j\to-\infty} P_j = \mu_p$ then

$$\begin{aligned} \liminf_{n\to\infty} d(P,P\circ T^{n}) \;\; &\geq \;\; d_N(P,Q) \\ &\geq \;\; M. \end{aligned}$$

Since M is arbitrary then

$$\lim_{n\to\infty} d\left(P,P\circ T^{n}\right) = \infty.$$

$\square$

Lemma 5. *Let P be a non-singular product measure on $\{0,1\}^{\mathbb{Z}}$ such that*

$$\liminf_{k\to-\infty} P_k\left(\{0\}\right) \neq \limsup_{k\to-\infty} P_k\left(\{0\}\right).$$

Then $(\mathbb{X},\mathcal{B}(\mathbb{X}),P,T)$ is NS zero-type.

Proof. Write $q_1 = \liminf_{k\to-\infty} P_k(\{0\})$ and $q_2 = \limsup_{k\to-\infty} P_k(\{0\})$.

Let $M>0$. Set $\alpha = \frac{q_2-q_1}{4}$. Define

$$A_{q_i} := \left\{n\in\mathbb{Z}:\ \left|P_k(\{0\})-q_i\right| < \alpha\right\},\ i=1,2.$$

Let $A_{q_i}^N = A_{q_i}\cap[-N,N]$.

Choose N large enough so that

$$\left|A_q^N\right| \geq \frac{M}{\alpha} \ \text{ and } \ \left|A_p^N\right| \geq \frac{M}{\alpha}.$$

Since $d(P,P\circ T)<\infty$ then for every $j\in\mathbb{Z}\cap[-N,N]$,

$$\lim_{n\to\infty}\left|P_{-n}(\{0\}) - P_{-n+j}(\{0\})\right| = 0.$$

Therefore for large enough $n\in\mathbb{N}$ either

$$[-N-n, N-n]\cap A_{q_1} = \emptyset$$

or

$$[-N-n, N-n] \cap A_{q_2} = \emptyset.$$

Therefore for large enough $n \in \mathbb{N}$,

$$\begin{aligned} d\left(P, P \circ T^n\right) &\geq d_N(P,Q) \\ &\geq \sum_{k \in A_{q_1}^N} \left(\sqrt{P_k\left(\{0\}\right)} - \sqrt{P_{k-n}\left(\{0\}\right)}\right)^2 \\ &\quad + \sum_{k \in A_{q_2}^N} \left(\sqrt{P_k\left(\{0\}\right)} - \sqrt{P_{k-n}\left(\{0\}\right)}\right)^2 \\ &\geq \min\left(\alpha \cdot \left|A_{q_i}^N\right|, \alpha \cdot \left|A_{q_2}^N\right|\right) \geq M. \end{aligned}$$

Therefore

$$\lim_{n \to \infty} d\left(P, P \circ T^n\right) = \infty$$

and the shift is NS zero-type.

□

2. A zero type and power weakly mixing Bernoulli shift

In this section we construct a Bernoulli shift which is zero type and power weak mixing. The construction is done by imposing a stronger growth condition on the shift constructed in [Kos].

Construction of the product measure:

The product measure will be $P = \prod_{k=-\infty}^{\infty} P_k$, where

$$\forall i \geq 0,\ P_i\left(0\right) = P_i\left(1\right) = \frac{1}{2}. \tag{2.1}$$

The definition of P_k for negative k's is more complicated as it involves an inductive procedure.

2.1. The inductive definition of P_k for negative $k's$.

We will need to define inductively 5 sequences $\{\lambda_t\}_{t=1}^{\infty}, \{n_t\}_{t=1}^{\infty}$, $\{m_t\}_{t=1}^{\infty}$, $\{M_t\}_{t=0}^{\infty}$ and $\{N_t\}_{t=1}^{\infty}$. The sequence $\{\lambda_t\}$ is of real numbers which decreases to 1. The other four, $\{n_t\}_{t=1}^{\infty}$, $\{m_t\}_{t=1}^{\infty}$, $\{M_t\}_{t=0}^{\infty}$ and $\{N_t\}_{t=1}^{\infty}$ are increasing sequences of integers.

First choose a positive summable sequence $\{\epsilon_t\}_{t=1}^{\infty}$ and set $M_0 = 1$.

Base of the induction: Set $\lambda_1 = 2$, $n_1 = 2$, $m_1 = 4$. Set also $N_1 = M_0 + n_1 = 3$ and $M_1 = N_1 + m_1 = 7$.

Given $\{\lambda_u, n_u, N_u, m_u, M_u\}_{u=1}^{t-1}$, we will choose the next level $\{\lambda_t, n_t, N_t, m_t, M_t\}$ in the following order. First we choose λ_t depending on M_{t-1} and ϵ_t. Given λ_t we will choose n_t and then N_t will be defined by

$$N_t := M_{t-1} + n_t.$$

Then given N_t we will choose m_t and finally set

$$M_t := N_t + m_t.$$

Choice of λ_t: Set $k_t := \left\lfloor \log_2 \left(\frac{M_{t-1}}{\epsilon_t} \right) \right\rfloor + 1$, where $\lfloor x \rfloor$ denotes the integral part of x. Then set $\lambda_t = e^{\frac{1}{2^{k_t}}}$. With this choice of λ_t we have,

$$\lambda_t^{M_{t-1}} < e^{\epsilon_t}. \tag{2.2}$$

This choice of λ_t has the property that for every $u < t$, $\lambda_u = \lambda_t^{2^{k_t - k_u}}$.

Define

$$A_{t-1} := \left\{ \prod_{u=1}^{t-1} \lambda_u^{x_u} \; : \; x_u \in [-n_u, n_u] \right\}.$$

Choice of n_t: Given $\{\lambda_u, n_u, m_u\}_{u=1}^{t-1}$ and λ_t, the set A_{t-1} is a finite subset of $\lambda_t^{\mathbb{Z}}$. Choose n_t which satisfies

$$\lambda_t^{n_t/4} \geq \max \left\{ a^2 : a \in A_{t-1} \right\}.$$

Choice of m_t: Now that n_t is chosen we set $N_t = M_{t-1} + n_t$. Set

$$tN_t \left(2 + 2^{tN_t} \right) = m_t. \tag{2.3}$$

Remark. Since m_t satisfies 2.3 then for every $k < t$ and $n \leq N_t$,

$$\frac{m_t}{n} - N_t > 2^{kN_t} \tag{2.4}$$

Definition 6. Let (X, μ, T) be a non singular automorphism such that $\mathcal{B}(X) \neq \{\emptyset, X\}$ and let $\mathcal{F} \subset \mathcal{B}$ be a factor algebra. Then:

(i) $\mathcal{F}$ is *exhaustive* if $\vee_{n=0}^{\infty} T^n \mathcal{F} = \mathcal{B}$.

(ii) $\mathcal{F}$ is *exact* if $\cap_{n=0}^{\infty} T^n \mathcal{F} = \{\emptyset, X\}$.

(iii) T is *a K-automorphism* if it is conservative and admits a factor algebra $\mathcal{F} \subset \mathcal{B}$ that is exhaustive and exact and such that T' is $\mathcal{F}$ measurable.

Remark. Krengel has shown in [Kre, p. 153-154] that all K-automorphisms are ergodic. See also [ST, Proposition 4.8(a)].

Theorem 7. *The Bernoulli shift* $(\mathbb{X}, \mathcal{B}(\mathbb{X}), P, T)$ *is a non-singular, type* III_1, *zero-type and power weakly mixing transformation.*

Proof. In [Kos] it is shown that the shift is a type III_1 transformation. Therefore by Theorem 3 the shift is a zero type transformation. It remains to show that the shift is power weak mixing.

Let $l_1, l_2, ..., l_k \in \mathbb{Z}\backslash\{0\}$ and denote by $S := T^{l_1} \times T^{l_2} \times \cdots \times T^{l_k}$. Clearly

$$S^{n'}(w_1, w_2, .., w_k) \quad = \quad \prod_{i=1}^{k} T^{(l_i n)'}(w_i)$$

and $\left(\mathbb{X}^k, P^{\times k}, S\right)$ admits an exhaustive and exact factor. Therefore in order to prove the ergodicity of S it is sufficient to show that S is conservative.

By [Kos, lemma 3] and a similar calculation for negative $n's$ there exists $t_0 \in \mathbb{N}$ such that for every $t > t_0$, $|n| \in [N_t, m_t)$ and $w \in \mathbb{X}$

$$T^{n'}(w) \geq \sqrt[k]{\frac{1}{2}} \prod_{u=1}^{t} \lambda_u^{\sum_{j=-N_u+1}^{-M_{u-1}} \{w_{k+n} - w_k\}} \geq 2^{-N_t - 1/k}.$$

Here the last inequality follows from $\lambda_u < \lambda_1 = 2$.

Let $L = \max\{|l_i| : 1 \leq i \leq k\}$ .Then for every $t > \max(t_0, L)$, $i \in \{1, .., k\}$ and $N_t \leq n \leq \frac{m_t}{L}$,

$$T^{(l_i n)'}(w) \geq 2^{-N_t - 1/k}$$

and so

$$S^{n'}(w_1, w_2, .., w_k) \geq 2^{-kN_t - 1}.$$

Which together with (2.4) implies that every $\tilde{w} \in \mathbb{X}^k$ and $t > \max(t_0, k, L)$

$$\sum_{n=N_t}^{m_t/L} S^{n'}(\tilde{w}) \geq \left(\frac{m_t}{L} - N_t\right) 2^{-kN_t - 1} \geq \frac{1}{2}.$$

Therefore for every $\tilde{w} \in \mathbb{X}^k$,

$$\sum_{n=1}^{\infty} S^{n'}(\tilde{w}) \geq \sum_{t} \sum_{n=N_t}^{m_t/L} S^{n'}(\tilde{w}) = \infty.$$

By Hopf's theorem for non-singular transformations S is conservative.

$\square$

The next example is a continuous time flow such that all the times are zero type and power weakly mixing. In this Markov Chain example the flow preserves an infinite measure.

2.2. **The Markov Chain Example.**

A Borel map $X \times \mathbb{R} \ni (x, t) \mapsto \phi_t(x)$ such that

$$\phi_t \phi_s = \phi_{t+s}$$

is called a *non-singular flow* on $(X,\mathcal{B})$. Given a measure μ on X, the semi-flow $\{\phi_t\}_{t\in[0,\infty)}$ is called exact if

$$\cap_{t\geq 0}\phi_t^{-1}\mathcal{B}=\{\emptyset,X\}\, mod\mu.$$

A measure preserving flow $(X,\mathcal{B},\mu,\{\phi_t\})$ is a *K-flow* if it admits an exhaustive and exact factor. Clearly a natural extension of an exact semiflow is a K-flow.

The flow is called *Power Weakly Mixing* if for every $t_1,t_2,..,t_n\in\mathbb{R}$,

$$\phi_{t_1}\times\phi_{t_2}\times\cdots\phi_{t_n}$$

is an ergodic transformation of $\left(\left(S^{\mathbb{R}}\right)^k,\mathcal{B}^{\otimes k},\mu^{\otimes k}\right)$.

A function $p:[0,\infty)\to[0,1]$ is a *Markovian Renewal Function* if there exists a countable state space, which will be denoted by S, Markov Chain $\{X_t\}_{t\in[0,\infty)}$ and a state $a\in S$ such that

$$P_{a,a}(t):=P\left(X_t=a|\,X_0=a\right)=p(t).$$

We say that p is *aperiodic* if

$$\gcd\{n\in\mathbb{N}:\ p(n)\neq 0\}=1,$$

and *null recurrent* if

$$\sum_{n=1}^{\infty}p(n)=\infty\ and\ p(n)\xrightarrow[n\to\infty]{}0.$$

Given a renewal function p the sequence $\{p(n)\}_{n\in\mathbb{N}}$ defines a renewal sequence for the discrete time Markov Chain $\{X_n\}_{n=0}^{\infty}$. Thus if p is aperiodic and null recurrent then the Markov chain $\{X_n\}_{n=0}^{\infty}$ is aperiodic and null recurrent. Hence there exists a stationary (σ-finite) measure $\tilde{\mu}\in\mathcal{M}(S)$. It follows that the measure

$$\mu(\{a\})=\int_0^1\left(\sum_{s\in S}P_{s,a}(t)\tilde{\mu}(\{s\})\right)dt$$

is a stationary measure for $(P_t)_{t\in\mathbb{R}}=\left(\{P_{a_1,a_2}(t)\}_{a_1,a_2\in S}\right)_{t\in\mathbb{R}}$. Finally let $\nu=P^{\mu}$ be the measure on $S^{\mathbb{R}}$ with finite dimensional distributions

$$\nu\left[x_{t_0}=s_0,x_{t_1}=s_1,...,x_{t_n}=s_n\right]=\mu\left(\{s_0\}\right)P_{s_0,s_1}\left(t_1-t_0\right)\cdots P_{s_{n-1},s_n}\left(t_n-t_{n-1}\right),$$

for every $t_0<t_1<\cdots<t_n$ and $s_0,s_1,\ldots,s_n\in S$. The flow $\{\phi_t\}_{t\in\mathbb{R}}$ on $S^{\mathbb{R}}$ defined by

$$\phi_t w(s)=w(s+t)$$

is ν- measure preserving. It is the natural extension of the semiflow $\{\phi_t\}_{t\in[0,\infty)}$.

Theorem 8. *Let $p:[0,\infty)\to[0,1]$ be an aperiodic and null recurrent Markov Renewal Function then the flow $\left(S^{\mathbb{R}},\mathcal{B},\nu,\{\phi_t\}_{t\in\mathbb{R}}\right)$ is conservative, exact and zero-type. If in addition for every $t_1,t_2,..,t_n\in\mathbb{R}_+$,*

$$\sum_{n=1}^{\infty}\prod_{j=1}^{k}p_{a,a}^{(n)}(t_j)=\infty$$

then the flow is Power Weakly Mixing.

Proof. Since $\{X_t\}_{t\geq 0}$ is a null recurrent Markov chain the flow is conservative and zero-type.

First we show that the tail σ-field of $\{X_t\}_{t\geq 0}$ is trivial, hence the semiflow $\{\phi_t\}_{t\in[0,\infty)}$ is exact. Let $h>0$ and observe that $\Upsilon=\left(S^{h\mathbb{N}},\mathcal{B}_{S^{h\mathbb{N}}},\nu|_{h\mathbb{N}},\phi_h\right)$ is a factor of $\mathfrak{X}=\left(S^{[0,\infty)},\mathcal{B},\nu,\phi_h\right)$. Since the discrete time chain $\{X_{nh}\}_{n\in\mathbb{N}}$ is aperiodic and recurrent, it follows by [BF] that its tail σ-algebra is trivial.

Denote by $\mathcal{F}_h=\mathcal{B}\cap S^{[0,h]}$. Then it follows from the Markov property that given Υ, for every $\Lambda_0,\Lambda_1,...,\Lambda_n\in\mathcal{F}_h$, the sets

$$[w\in\Lambda_1],[\phi_h w\in\Lambda_2],...,[\phi_{nh}w\in\Lambda_n]$$

are independent. By Kolmogorov's zero-one law $\mathfrak{X}$ is an exact non-singular extension of Υ in the sense of [AD]. Therefore, since Υ is exact, it follows by Proposition 4 in [AD] that $\mathfrak{X}$ is exact. See also [Ios, Theorem 6]. Therefore since the flow $\{\phi_t\}_{t\in\mathbb{R}}$ is the natural extension of the semiflow, it is a K-flow.

It follows that for every $t_1,t_2,..,t_k\in\mathbb{R}$ the transformation $R=\phi_{t_1}\times\phi_{t_2}\times\cdots\phi_{t_n}$ is K and in order to prove ergodicity of R it is enough to show conservativity which is a consequence of the fact that,

$$\sum_{n=1}^{\infty}\prod_{j=1}^{k}p_{a,a}^{(n)}\left(|t_j|\right) = \infty.$$

□

Example: Let

$$p(t)=\frac{1}{\log(e+t)}.$$

Since $p(t)$ satisfies the conditions of [Kin, Theorem 6.6, p.144, see also p.41] there exists a continuous time Markov Chain $\{X_t:t\in\mathbb{R}\}$ on a countable state space S such that for

some $a \in S$,

$$p_{a,a}(t) := P\left(X_t = a \middle| X_o = a\right) = p(t).$$

Since

$$p(n) = \frac{1}{\log(e+n)} \neq 0,\ \sum_{n=1}^{\infty} p(n) = \infty$$

and for every $t_1, t_2, .., t_n > 0$,

$$\sum_{n=1}^{\infty} \prod_{j=1}^{k} p_{a,a}^{(n)}(t_j) = \sum_{n=1}^{\infty} \prod_{j=1}^{k} p_{a,a}(n \cdot t_j) = \sum_{n=1}^{\infty} \prod_{j=1}^{k} \frac{1}{\log\left(e + t_j \cdot n\right)} = \infty,$$

it satisfies the conditions of Theorem 8 and hence the Markov flow defined by the Markov Chain is conservative, zero type and power weakly mixing.

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